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Finite-size scaling investigations in the quantum φ^4-model with long-range interaction

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Abstract. In this paper, we study in detail the critical behaviour of the $\mathcal{O}(n)$ quantum φ^4-model with long-range interaction decaying with distances r by a power law as $r^{-d-\sigma}$ in the large-n limit. The zero-temperature critical behaviour is discussed. Its alteration by finite temperature and/or finite sizes in the space is studied. The scaling behaviours are studied in different regimes depending upon whether the finite temperature or the finite sizes of the system lead. A number of results for the correlation length, critical amplitudes and the finite-size shift, for different dimensionalities between the lower $d_< = \sigma/2$ and the upper $d_> = 3\sigma/2$ critical dimensions, are calculated.

1. Introduction

The vast majority of existing analytical and numerical work on finite-size effects has been in regimes where the quantum effects are entirely ruled out [1]. In this case both the statics and the dynamics can be described by classical statistical models. At the critical point the bulk thermodynamic functions of these models are singular. For these systems the finite-size scaling theory asserts that the singularities holding at the thermodynamic limit are altered depending upon the nature of the geometry to which the system is confined and the imposed boundary conditions [2, 3]. Indeed, various types of geometries can be considered, depending on the number of the finite sizes in the model.

The $\mathcal{O}(n)$ vector models are extensively used to explore the finite-size scaling theory, using various methods and techniques. For finite n the most frequently used method is that of the renormalization group [4]. The most thoroughly investigated case is the one corresponding to the limit $n = \infty$ (this limit also includes the mean spherical model) [3]. In this limit, these models are exactly soluble for arbitrary dimensions and in a general geometry. The majority of these investigations are limited to systems in which the forces are of short range. To test the finite-size scaling idea when the interaction is of long range (varying with a power law), the only model used is the mean spherical model [5].

In recent years there has been increased interest in the theory of zero-temperature quantum phase transitions [6, 7]. Distinct from temperature-driven critical phenomena, these phase transitions occur at zero temperature as a function of some non-thermal control parameter (or as a competition between different parameters describing the basic interaction of the system), and the relevant fluctuations are of quantum rather than thermal nature. In these types of critical phenomena time plays a crucial and fundamental role. The coupling of statistics and dynamics that is inherent to quantum statistical problems introduces effective dimensionalities in the hyperscaling laws, i.e. the space dimensionality d is replaced by $d + z$ (z is the dynamic

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critical exponent). In this case the inverse temperature acts as a finite size in the 'imaginary-time' direction for the quantum system at its critical point. This allows the investigation of scaling laws for quantum systems near the quantum critical point in terms of the theory of finite-size scaling [8] or using the conformal field theory techniques by mapping the bulk system in a finite one [9]. The $\mathcal{O}(n)$ symmetric vector models are also used in the exploration of quantum critical phenomena and for the investigation of the scaling properties of such phase transitions. The quantization of classical $\mathcal{O}(n)$ is performed with the help of the Trotter formula which maps the quantum model on a classical one with z additional effective dimensions.

In systems showing quantum critical behaviour the temperature plays two different roles. For temperatures low enough, quantum effects are essential. In this case the temperature affects the geometry to which the system is confined adding 'new' sizes to the Euclidean space–time coordinate system. By raising the temperature, the system is driven away from quantum criticality. At high temperatures, however, the size in the 'imaginary-time' direction becomes irrelevant in comparison with all length scales in the system. In this case we have a classical system in d dimensions and the temperature is just a coupling constant in the classical critical behaviour.

In this paper we present a detailed investigation of the scaling properties of the quantum $\mathcal{O}(n)$ vector φ^4-model with long-range interaction. Our study will include the quantum as well as the finite-size effects and their influence on the critical behaviour and the critical amplitudes. We will also check the influence of the interaction range on the critical behaviour. These interactions enter the exact expressions for the free energy only through their Fourier transform in which the leading asymptotic is $U(q) \sim q^{\sigma^*}$, where $\sigma^* = \min(\sigma, 2)$ [10]. As was shown for bulk systems by renormalization group arguments, $\sigma \geqslant 2$ corresponds to the case of finite(short)-range interactions, i.e. the universality class then does not depend on σ [10]. Values satisfying $0 < \sigma < 2$ correspond to long-range interactions and the critical behaviour depends on σ. Following the above reasoning one usually considers the case $\sigma > 2$ as uninteresting for critical effects, even for the finite-size treatments [11]. So, here we will consider only the case $0 < \sigma \leqslant 2$.

The paper is devised as follows. In section 2 we present some predictions which extend the finite-size scaling to quantum systems. We also discuss the interplay of quantum and finite-size effects on the quantum critical behaviour. We comment on the anisotropy caused by the presence of both effects. In section 3 we review, briefly, the φ^4-model with long-range interaction and present the saddle point equation. We investigate the low-temperature behaviour of the bulk model in section 4. In section 5, we investigate in detail the finite-size behaviour at zero temperature. Section 6 is devoted to some comments about the low-temperature and finite-size effects of the system. In section 7 we discuss our result briefly. In the remainder of the paper we present some details of the calculations.

2. Finite-size scaling and quantum critical behaviour

Divergent length scales play a crucial role in continuous phase transitions. Unlike classical models, where the scaling can be done equally for all 'spatial' dimensions, quantum models are anisotropic in general, and therefore the 'space' and 'imaginary-time' directions will not scale in the same fashion. According to the general hypothesis of finite-size scaling theory [12] extended here for a quantum (anisotropic) system, a physical quantity $\mathcal{A}(r, h, L, T)$ (where r is the distance from the critical point, h is an ordering field coupled to the order parameter, T is the temperature and L is the size of the system), which may be singular at the critical point in the thermodynamic (bulk) limit at the quantum critical point ($r = 0$), will scale as

$$\mathcal{A}(r, h, T, L) = b^p \mathcal{A}_s(rb^{1/\nu}, hb^{\Delta/\nu}, Tb^z, bL^{-1}). \tag{2.1}$$

In the scaling form (2.1), p corresponds to the engineering dimension $d+z$ of the system in the case where the scaling function refers to the singular part of the free energy, and it is then divided by ν, the critical exponent measuring the divergence of the bulk thermodynamic function $\mathcal{A}$ at the critical point for the other physical quantities of interest (for the correlation length $p = 1$, for the susceptibility $p = \gamma/\nu$, etc). Depending upon the choice of the renormalization group rescaling factor b we obtain different scaling functions $\mathcal{A}_s$, which are related to each other by some appropriate change of the scaling variables.

Before starting to discuss the general form of the scaling universal function $\mathcal{A}_s$, we will discuss the two limiting cases which were subjects of several investigations during the past two decades.

The first case corresponds to zero temperature and is called, hereafter, the quantum critical behaviour. Here, equation (2.1) reduces to

$$\mathcal{A}(r, h, 0, L) = b^p \mathcal{A}_s^L(rb^{1/\nu}, hb^{\Delta/\nu}, bL^{-1}). \tag{2.2}$$

Following [13], we choose the rescaling factor b to be proportional to the linear size of the system. Then, we obtain

$$\mathcal{A}(r, h, 0, L) = L^p \mathcal{A}_s^L(rL^{1/\nu}, hL^{\Delta/\nu}). \tag{2.3}$$

Here the situation resembles that of systems exhibiting classical (thermal) phase transition.

It is possible to get another result for the scaling function in the right-hand side of equation (2.3) if one considers the variable $\tilde{r}L^{1/\nu}$ instead of the first variable of the scaling function $\mathcal{A}_s^L$. The parameter $\tilde{r}$ is introduced in such a way so as to account for the shift of the critical quantum parameter to the value corresponding to the rounding of the thermodynamic functions, when the number of infinite dimensions d' is less than its lower critical dimension $d_<$. For the opposite case we have just a shift of the critical control parameter. In this case we find that the critical exponents are those of a d'-dimensional system (see, for example, [2]). By definition the finite-size shift is given by

$$\tilde{r} = r + \epsilon(L) \qquad \lim_{L\to\infty} \epsilon(L) = 0. \tag{2.4}$$

In general we have

$$\epsilon(L) \sim L^{-1/\nu} \tag{2.5}$$

and so this quantity shrinks to zero in the thermodynamic limit. When the arguments of the scaling functions get replaced by zero, we will obtain universal critical amplitudes, characterizing the whole class of universality. Here we have to emphasize that the values of the universal critical amplitudes are different depending upon the point in which they are calculated, i.e. r or $\tilde{r}$.

The second case we consider is the one corresponding to the bulk system ($L = \infty$) at finite temperature. In this case the scaling form (2.1) transforms into (see [7] and references therein)

$$\mathcal{A}(r, h, T, \infty) = T^{-p/z} \mathcal{A}_s^\tau(rT^{-1/z\nu}, hT^{-\Delta/z\nu}) \tag{2.6}$$

where we used the relation $b = T^z$ between the temperature and the rescaling parameter b. The same predictions for the shift and the critical amplitudes remain valid here. We find it convenient to use $L_\tau \sim T^{-z}$ as a linear 'temporal' size instead of the inverse temperature.

The general case corresponding to both quantum and finite-size effects can be studied by considering a system with larger dimension, $p = d + z$, confined to a general geometry of the form $L^{p-z} \times L_\tau^z$ [14]. The fact that the inverse temperature can be used as an additional size in the imaginary-time direction creates some anisotropy in the system. This property will lead to the establishment of some change in the scaling properties of the finite quantum system. In the general case we can consider the quantum-to-classical and the finite-size to the bulk system different crossover phenomena. It is easy to convince oneself about this statement by considering the dynamic critical exponent which is different for different quantum systems. The situation of the combined investigations of finite-size scaling and quantum-to-classical crossover is similar to the one formulated in the framework of the phenomenological study of finite-size scaling in anisotropic systems [15]. In this case one can investigate the interplay of quantum and finite-size scaling. For example, in the case when the quantum effects are leading compared with the finite-size effects (here this case will be called the 'low-temperature' case), i.e. $L_\tau \ll L$, from equation (2.1) to first order in L_τ/L we expect to obtain

$$\mathcal{A}(r,h,T,L) = L_\tau^p \mathcal{A}_s^\tau(rL_\tau^{1/\nu}, hL_\tau^{\Delta/\nu}) + L_\tau^{p+1}L^{-1}\mathcal{A}_s^{\tau L}(rL_\tau^{1/\nu}, hL_\tau^{\Delta/\nu}) \tag{2.7}$$

instead of equation (2.6). This shows how the finite-size effects give rise to some corrections to the quantum scaling. Following the same reasoning in the case of 'very low temperature' i.e. $L_\tau \gg L$, when the finite-size contributions to the ground state energy and its derivatives are dominant compared with those coming from the quantum effects, we find

$$\mathcal{A}(r,h,T,L) = L^p \mathcal{A}_s^L(rL_\tau^{1/\nu}, hL_\tau^{\Delta/\nu}) + L_\tau^{-z}L^{p+z}\mathcal{A}_s^{L\tau}(rL_\tau^{1/\nu}, hL_\tau^{\Delta/\nu}) \tag{2.8}$$

showing a correction to the zero-temperature finite-size scaling due to the temperature. These ideas have been tested in the framework of the spherical quantum rotor model in [16, 17]. We will compare the scaling forms (2.7) and (2.8) with the available analytical results obtained below.

3. The free energy and the gap equation

The quantum φ^4-model for which we are going to investigate its quantum critical and finite-size scaling properties is (for a review of the applicability of this model in exploring quantum critical phenomena, see [7])

$$\mathcal{H}\{\varphi\} = \frac{1}{2}\int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}x \left[(\partial_\tau\varphi)^2 + (\nabla^{\sigma/2}\varphi)^2 + r_0\varphi^2 + \frac{u_0}{2}\varphi^4\right] \tag{3.1}$$

where φ is a shorthand notation for the space–time-dependent n-component field $\varphi(x,\tau)$, u_0 and r_0 are model constants and T is the temperature. In (3.1) we assumed $\hbar = k_B = 1$ and the size scale is measured in units in which the velocity of excitations $c = 1$. Here we will consider periodic boundary conditions. This means

$$\varphi(x,\tau) = \sqrt{\frac{T}{V}} \sum_{\boldsymbol{k},\omega_l} \varphi(\boldsymbol{k},\omega_l) \exp(\mathrm{i}\boldsymbol{k}\cdot x - \mathrm{i}\omega_l\tau) \tag{3.2}$$

where $\omega_l = 2\pi T l$ (with $l = 0, \pm1, \pm2, \ldots$) are the Matsubara frequencies for bosonic systems, $\boldsymbol{k}$ is a discrete vector with components $k_i = 2\pi n_i/L$, $n_i = 0, \pm1, \pm2, \ldots$ and a cutoff Λ, and $V = L^d$ is the volume of the system. We note that the second term in the model transforms into $|\boldsymbol{k}|^\sigma\varphi^2(\boldsymbol{k},\omega)$ in the momentum representation, where the parameter $0 < \sigma \leqslant 2$ accounts for short-range and long-range interaction as well.

The partition function of the Hamiltonian (3.1) reads

$$\mathcal{Z} = \int \mathcal{D}\varphi \exp(-\mathcal{H}\{\varphi\}). \tag{3.3}$$

Note that in the low-temperature limit, $T \ll \Lambda$, the integral over τ in $\mathcal{H}\{\phi\}$ (see (3.1)) can be extended over the whole temperature axis to give an effective φ^4-model in $d+z$ dimensions with a quantum control parameter r_0. In the high-temperature limit $T \gg \Lambda$ the upper limit in the integral over τ is small. This offers us the possibility to write the Hamiltonian as a classical φ^4-model in d dimensions. Now using a standard decoupling procedure based on the Hubbard–Stratonovich transformation in (3.3), which introduces an auxiliary field ψ, one gets

$$\mathcal{Z} = \mathcal{C}\int \mathcal{D}\psi\mathcal{D}\varphi \exp\left\{-\frac{1}{2}\int_0^{1/T} \mathrm{d}\tau \int \mathrm{d}x\left[(\partial_\tau\varphi)^2 + (\nabla^{\sigma/2}\varphi)^2 + r_0\varphi^2 + \psi\varphi^2 - \frac{1}{2u_0}\psi^2\right]\right\}. \tag{3.4}$$

Using the fact that the field φ is n-component we decompose the integral over φ into an n-dimensional Gaussian integral, which can be performed easily, leading to

$$\mathcal{Z} = \mathcal{C}\int \mathcal{D}\psi \exp\left[\frac{n\beta V}{4u_0}\psi^2 - \frac{n}{2}\,\mathrm{Tr}\ln[r_0 + \psi - \partial_\tau^2 - \nabla^\sigma]\right]. \tag{3.5}$$

In the last expression we assumed that the field ψ is time- and space-independent. For large n we use the saddle point method to evaluate the integral over ψ in (3.5). Finally, we obtain the free energy per particle $\mathcal{F} = -(T/V)\,\mathrm{Tr}\ln\mathcal{Z}$ in the momentum space

$$\mathcal{F} = -\frac{1}{4u_0}(\phi - r_0)^2 + \frac{1}{V}\sum_{\boldsymbol{k}} \ln 2\sinh\left[\frac{1}{2T}\sqrt{\phi + |\boldsymbol{k}|^\sigma}\right] \tag{3.6a}$$

and the saddle point equation

$$\phi = r_0 + u_0\frac{T}{V}\sum_{\boldsymbol{k},m}\frac{1}{\phi + (2\pi mT)^2 + |\boldsymbol{k}|^\sigma} \tag{3.6b}$$

where, for convenience, we used the shifted parameter $\phi \equiv \psi + r_0$ instead of ψ itself.

Equations (3.6) are our starting point to explore the finite-size and quantum effects on the bulk critical behaviour of the model (3.1). Let us note that in the particular case of short-range forces $\sigma = 2$, we recover the results of [18] utilized to investigate the finite-size scaling when the quantum effects are absent. When the quantum effects are relevant, equation (3.6*b*) was used in [19] in order explore the quantum critical behaviour and to calculate the critical exponents.

To extract the physics from model (3.1), we calculate the susceptibility χ. In the large-n limit, we obtain

$$\chi \equiv \int \mathrm{d}^d x\,\langle\varphi(x)\varphi(0)\rangle = \frac{1}{\phi}. \tag{3.7}$$

This is a particular case of the general correlator $\chi(\boldsymbol{k},\omega) = \langle\phi(\boldsymbol{k},\omega_n)\phi(-\boldsymbol{k},-\omega_n)\rangle$, defining the dynamic susceptibility:

$$\chi(\boldsymbol{k},\omega_n) = (\phi + \omega_n^2 + |\boldsymbol{k}|^\sigma)^{-1}. \tag{3.8}$$

From this expression one can deduce that the anomalous critical exponent measuring the divergence of the correlation length at the quantum critical point is $\eta = 2 - \sigma$ and the dynamic critical exponent is $z = \sigma/2$. The denominator of the dynamic susceptibility (3.8) has poles in the complex $\boldsymbol{k}$-plane at $k = \pm(-\phi - (2\pi nT)^2)^{1/\sigma}$. The closest pole to the origin is the one corresponding to $n = 0$. This pole determines an exponential decay of the correlation functions. So the correlation length turns out to be

$$\xi = \phi^{-1/\sigma}. \tag{3.9}$$

From (3.7) and (3.9) one can deduce the simple ratio between the critical exponents γ and ν to be equal to σ.

To investigate the bulk quantum critical behaviour of the model (3.1) at zero temperature we transform the sums in equations (3.6) into integrals, i.e.

$$T\sum_{\omega_n} \to \int \frac{\mathrm{d}\omega}{2\pi} \qquad \text{and} \qquad \frac{1}{V}\sum_{\boldsymbol{k}} \to \int \frac{\mathrm{d}^k \boldsymbol{k}}{(2\pi)^d}$$

to get

$$\mathcal{F}_0 = -\frac{1}{4u_0}(\phi - r_0)^2 + \int \frac{\mathrm{d}\omega}{4\pi}\int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \ln[\phi + \omega^2 + |\boldsymbol{k}|^\sigma] \tag{3.10a}$$

and

$$\phi = r_0 + u_0 \int \frac{\mathrm{d}\omega}{2\pi}\int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} \frac{1}{\phi + \omega^2 + |\boldsymbol{k}|^\sigma}. \tag{3.10b}$$

These equations contain all the necessary information from which we can extract all that we need regarding the critical behaviour at zero temperature. Here we will pay attention to the scaling properties of the thermodynamic functions in the neighbourhood of the quantum critical point given by

$$r_{0c} = -\frac{u_0}{2}\int \frac{\mathrm{d}^d \boldsymbol{k}}{(2\pi)^d} |\boldsymbol{k}|^{-\sigma/2}. \tag{3.11}$$

This integral is infrared convergent only for $d > d_<$, where $d_< = \sigma/2$ defines the lower critical dimension. It is easy to calculate all the critical exponents for the model we are considering here in the large-n limit by expanding the right-hand side of equation (3.10b) for small ϕ. In the expanded expression one can see a natural emergence of $d_> = 3\sigma/2$ as the upper critical dimension. Here we will give some of these critical exponents:

$$\gamma = \frac{\sigma}{d - d_<} \qquad \nu = \frac{1}{d - d_<}. \tag{3.12}$$

In the remainder of this paper we will investigate the effects of finite temperature and/or spatial sizes on the bulk zero-temperature critical behaviour.

4. Finite temperature effects on the quantum critical behaviour

At finite temperature we obtain correction terms to the right-hand side of equations (3.10). These are given by

$$\Delta_{\mathcal{F}}^{\tau}(T,\phi) = -\frac{k_d}{\sigma\sqrt{\pi}}\Gamma\left(\frac{d}{\sigma}\right)\phi^{\frac{d}{\sigma}+\frac{1}{2}} \sum_{m=1}^{\infty} \frac{K_{\frac{d}{\sigma}+\frac{1}{2}}(m\frac{\sqrt{\phi}}{T})}{(m\frac{\sqrt{\phi}}{2T})^{\frac{d}{\sigma}+\frac{1}{2}}} \tag{4.1a}$$

for the free energy and

$$\Delta_{\xi}^{\tau}(T,\phi) = \frac{2k_d}{\sigma\sqrt{\pi}}\Gamma\left(\frac{d}{\sigma}\right)\phi^{\frac{d}{\sigma}-\frac{1}{2}} \sum_{m=1}^{\infty} \frac{K_{\frac{d}{\sigma}-\frac{1}{2}}(m\frac{\sqrt{\phi}}{T})}{(m\frac{\sqrt{\phi}}{2T})^{\frac{d}{\sigma}-\frac{1}{2}}} \tag{4.1b}$$

for the saddle point equation. In expressions (4.1) we used the quantity $k_d^{-1} = \frac{1}{2}(4\pi)^{\frac{d}{2}}\Gamma(d/2)$ and $K_\nu(x)$ is the MacDonald function (second modified Bessel function).

Combining equations (3.10b) and (4.1b) we get, in the bulk limit corresponding to the geometry $\infty^d \times L_\tau^z$, the following scaling form (for the saddle point equation) for dimensions d between the lower $d_<$ and upper critical dimensions $d_>$:

$$x_\tau = u_0 k_d \frac{y_\tau^{d/z-1}}{2\sigma\sqrt{\pi}}\Gamma\left(\frac{d}{\sigma}\right)\left[\Gamma\left(\frac{1}{2}-\frac{d}{\sigma}\right) + 4\sum_{m=1}^{\infty}\frac{K_{\frac{d}{\sigma}-\frac{1}{2}}(my_\tau)}{(\frac{1}{2}my_\tau)^{\frac{d}{\sigma}-\frac{1}{2}}}\right]. \tag{4.2}$$

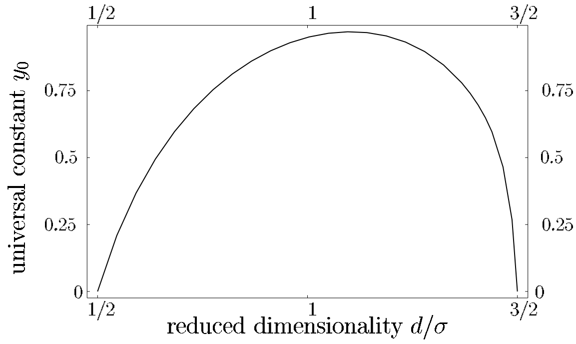

Figure 1. Solution of equation (4.2) at $x_\tau = 0$ as a function of the reduced space dimensionality of the system.

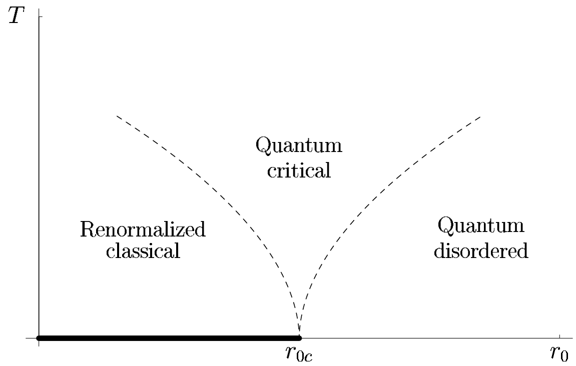

Figure 2. Qualitative phase diagram of the model under consideration in the vicinity of the quantum critical point for $d = \sigma$. The dashed curves are crossover lines. The thick line shows the region where the system is ordered at $T = 0$.

Here the scaling variables are defined by $y_\tau = \sqrt{\phi}/T$ and $x_\tau = T^{1-d/z}(r_{0c} - r_0)$.

For fixed σ, the solution, y_0, of (4.2) at the quantum critical point ($x_\tau = 0$) is a universal number. An analytic solution of this equation cannot be obtained in the general case. Here we will consider some particular cases:

$$y_0 = \begin{cases} \dfrac{2\pi}{\sigma}(d - d_<) & d - \sigma/2 \ll 1 \\ 2\ln\dfrac{1+\sqrt{5}}{2} & d = \sigma \\ 2\pi\sqrt{\dfrac{d_> - d}{3\sigma}} & 3\sigma/2 - d \ll 1. \end{cases} \tag{4.3}$$

At $x_\tau = 0$, equation (4.2) can be solved numerically. The behaviour of the universal constant y_0 as a function of the reduced dimensionality d/σ is given in figure 1. We see that y_0 depend upon the ratio d/σ in a universal way for all values of σ smaller than or equal to 2.

The phase diagram of the model (here we are concerned with the critical line and crossover lines) in the vicinity of the quantum critical point is determined by putting $\phi = 0$ in the saddle point equation (3.6*b*). In the vicinity of the quantum critical point (3.11) the phase diagram is determined by

$$r_{0c} - r_{0c}(T) = \frac{2}{\sigma}k_d T^{(2d/\sigma-1)}\Gamma\left(\frac{2d}{\sigma} - 1\right)\zeta\left(\frac{2d}{\sigma} - 1\right) \tag{4.4}$$

where $\zeta(x)$ is the Riemann zeta function. The crossover lines and the critical line are determined by $|r_{0c} - r_0| \sim T^{1/\nu z}$. The phase diagram for a particular case $d = \sigma$ is presented in figure 2. In the different regions of the phase diagram and for arbitrary dimensions the correlation length has different behaviour.

For $x_\tau \to \infty$ in the interval $\sigma/2 < d < \sigma$, the correlation length behaves like

$$\xi \sim \left|\frac{T}{r_0 - r_{0c}}\right|^{1/(d-\sigma)}. \tag{4.5}$$

This expression shows that, when $T \to 0^+$, the correlation length goes to infinity as quantum effects become relevant. In this case there is no quantum phase transition for finite temperature. For the second interval, $\sigma < d < 3\sigma/2$, one gets the behaviour

$$\xi \sim \left(\frac{T}{r_0 - r_{0c}(T)}\right)^{1/(d-\sigma)} \tag{4.6}$$

for r_0 very close to but larger than the critical quantum parameter $r_{0c}(T)$. Here we have a phase transition at finite temperature and the critical exponents are those of the model in the classical limit. They are

$$\gamma = \sigma(d-\sigma)^{-1} \qquad \nu = (d-\sigma)^{-1}. \tag{4.7}$$

For r_0 less than its critical value the correlation length is infinite.

For $x_\tau \to -\infty$ the correlation length is temperature independent. For arbitrary dimension $d_< < d < d_>$ it is given by

$$\xi \sim (r_{0c} - r_0)^{1/(d_< - d)}. \tag{4.8}$$

The particular case $d = \sigma$ is very simple to handle. In this case equation (4.2) becomes simpler. Its solution leads to

$$\xi^{-z} = T f_\xi(x_\tau) \tag{4.9}$$

for the inverse correlation length. Here

$$f_\xi(x_\tau) = 2\,\mathrm{arcsinh}\left[\frac{1}{2}\exp\left(-\frac{\sigma}{2}\frac{x_\tau}{k_\sigma}\right)\right] \tag{4.10}$$

is a scaling function, which simplifies in some limiting cases:

$$f_\xi(x) = \begin{cases} -\dfrac{\sigma x}{2k_\sigma} & x \to -\infty \\ \dfrac{1}{2}\ln\dfrac{1+\sqrt{5}}{2} & x = 0 \\ \exp\left(\dfrac{\sigma x}{2k_\sigma}\right) & x \to \infty. \end{cases} \tag{4.11}$$

From equations (4.11) one can transparently see the different behaviour of the correlation length $\xi(T)$ in three regions:

(a) *Renormalized classical region* with exponential divergence as $T \to 0$. In this region the system displays characteristics of the ordered ground state. The thermal fluctuations destroy long-range order at any finite temperature.
(b) *Quantum critical region* with $\xi(T) \sim T^{-2/\sigma}$ and crossover lines $T \sim |r_{0c} - r_0|^{\sigma/2}$. In this region the system 'notices' that it is finite in the time direction.
(c) *Quantum disordered region* with temperature-independent correlation length as $T \to 0$. In this region the system has a gap in the spectrum.

The different regions are qualitatively shown in the phase diagram in figure 2.

5. Quantum critical finite-size scaling

As the temperature is set to zero, equation (3.6) turns into

$$\mathcal{F}_L = \frac{1}{2V}\sum_{\boldsymbol{k}}\sqrt{\phi + |\boldsymbol{k}|^\sigma} - \frac{1}{4u_0}(\phi - r_0)^2 \tag{5.1a}$$

for the free energy and

$$\phi - r_0 = \frac{u_0}{2V}\sum_{\boldsymbol{k}}\frac{1}{\sqrt{\phi + |\boldsymbol{k}|^\sigma}} \tag{5.1b}$$

for the saddle point equation.

Exploring the finite-size scaling in the quantum limit ($T = 0$) turns out to be a difficult task because of the presence of the term $|\boldsymbol{k}|^\sigma$ in the spectrum of the model. In other words, the problem is how to handle the terms $\sqrt{\phi + |\boldsymbol{k}|^\sigma}$ in the expressions for the free energy and in that of the saddle point equation (5.1). The solution to this problem was given in [20], where two important identities facilitating the analysis of the finite-size scaling were presented (see also appendix A).

For the d-dimensional system with spatial geometry $L^{d-d'} \times \infty^{d'}$ at zero temperature and periodic boundary conditions imposed along the $(d - d')$ finite-size directions with linear size L of the system, the corrections to equations (5.1) are given by

$$\mathcal{F}_L = \frac{1}{2}\int\frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\sqrt{\phi + |\boldsymbol{k}|^\sigma} - \frac{1}{4u_0}(\phi - r_0)^2 - u_0L^{d+z}\Delta^L_{\mathcal{F}}(L^\sigma\phi) \tag{5.2a}$$

and

$$\phi = r_0 + \frac{u_0}{2}\int\frac{\mathrm{d}^d\boldsymbol{k}}{(2\pi)^d}\frac{1}{\sqrt{\phi + |\boldsymbol{k}|^\sigma}} + u_0L^{z-d}\Delta^L_{\xi}(L^\sigma\phi) \tag{5.2b}$$

where for convenience we introduced the following functions:

$$\Delta^L_{\mathcal{F}}(y) = \frac{\sigma}{8}\frac{1}{(4\pi)^{d/2}}\sum_{\boldsymbol{l}}{}'\int_0^\infty \mathrm{d}x\,\exp\left(\frac{\boldsymbol{l}^2}{4x}\right)x^{-\frac{\sigma}{4}-\frac{d}{2}-1}\mathcal{G}_{\frac{\sigma}{2},1-\frac{\sigma}{4}}(-x^{\frac{\sigma}{2}}y) \tag{5.2c}$$

and

$$\Delta^L_{\xi}(y) = \frac{1}{2}\frac{1}{(4\pi)^{d/2}}\sum_{\boldsymbol{l}}{}'\int_0^\infty \mathrm{d}x\,\exp\left(\frac{\boldsymbol{l}^2}{4x}\right)x^{\frac{\sigma}{4}-\frac{d}{2}-1}\mathcal{G}_{\frac{\sigma}{2},\frac{\sigma}{4}}(-x^{\frac{\sigma}{2}}y). \tag{5.2d}$$

Here the primed summation over the vector $\boldsymbol{l}$ is $(d - d')$-dimensional and the prime indicates that the term corresponding to $\boldsymbol{l} = 0$ is excluded. In the last equation we used the function [20]

$$\mathcal{G}_{\alpha,\beta}(x) = \frac{1}{\sqrt{\pi}}\sum_{k=0}^{\infty}\frac{\Gamma(k + 1/2)}{\Gamma(\alpha k + \beta)}\frac{x^k}{k!}. \tag{5.2e}$$

Some properties of the functions $\mathcal{G}_{\alpha,\beta}$ and $\Delta^L_\xi(y)$ are discussed in appendices A and B, respectively.

5.1. The finite-size shifted critical quantum parameter

It is well known from finite-size scaling theory that the critical value of the parameter driving the phase transition is shifted due the effects of the finite sizes in the system. The aim of this section is to evaluate the distance over which the critical quantum parameter r_{0c} of the bulk system is shifted. For our concrete model this is obtained by substituting the parameter ϕ by zero. The result is

$$r_{0c} - r_{0c}(L) = \frac{u_0}{2}\frac{\Gamma(d/2-\sigma/4)}{(4\pi)^{d/2}\Gamma(\sigma/4)}\sum_{l}{}' \left(\frac{|l|L}{2}\right)^{-d+\sigma/2} \tag{5.3}$$

and the $(d-d')$-dimensional sum in the right-hand side of equation (5.3) is convergent for $d' > d_<$.

In the opposite case $d_< > d'$, the right-hand side of (5.3) is divergent. Nevertheless, the sum in equation (5.3) can be expressed in terms of the Epstein zeta function, which is a generalization of the Riemann zeta function. The resulting Epstein function can be analytically continued beyond its domain of convergence to give a physical meaning to equation (5.3) as well. In this case the shifted 'pseudocritical' quantum parameter $r_{0c}(L)$ corresponds to the centre of the rounding of the singularities of the thermodynamic functions, taking place in the thermodynamic limit. This point has been investigated in detail in the framework of the finite-size shift of the critical temperature for the spherical model in [21].

Let us also mention that the distance over which the critical quantum parameter is shifted can be also expressed as

$$r_{0c} - r_0(L) = \frac{u_0}{2(2\pi)^{\sigma/2}}\frac{\pi^{d'/2}}{\Gamma(\sigma/4)}\mathcal{C}_{d,d',\sigma} \tag{5.4}$$

where $\mathcal{C}_{d,d',\sigma}$ is a the Madelung-type constant (cf appendix B).

One can see that the shifted critical quantum parameter $r_{0c}(L)$ is lower than its bulk critical value r_{0c} for the different values of d, d' and σ (which is the 'normal case', see [21]), while the pseudocritical $r_{0c}(L)$ is higher than the bulk critical quantum parameter. However, for the boundary case when $d' \to d_<$, we find that the shift is infinite. This may be explained with the aid of the behaviour of the Epstein zeta function at its pole $d' = d_<$. The shift in this case is $\delta r_0 \sim (d'-d_<)^{-1}$ and the appearance of $\mp\infty$ is clear.

5.2. Finite-size scaling at zero temperature

In the neighbourhood of the quantum critical point, it is possible to write equations (5.2) in the scaling forms (for dimensions between the lower $d_<$ and the upper $d_>$ critical dimensions)

$$\mathcal{F}_L - \mathcal{F}_0 = L^{-d-z}\left[-D^{\mathcal{F}}_{d,\sigma} y_L^{\frac{d}{\sigma}+\frac{1}{2}} + \frac{1}{2u_0}x_L y_L - \Delta^L_{\mathcal{F}}(y_L)\right] \tag{5.5a}$$

for the singular part of the free energy and

$$x_L = \frac{u_0}{2}D_{d,\sigma} y_L^{\frac{d}{\sigma}-\frac{1}{2}} + u_0\Delta^L_{\xi}(y_L) \tag{5.5b}$$

for the saddle point equation, whose solution is related to the correlation length. Here the scaling variables are given by $x_L = (r_{0c} - r_0)L^{d-d_<}$ and $y_L = L^{\sigma}\phi$, and

$$D_{d,\sigma} = -2D^{\mathcal{F}}_{d,\sigma}\left(\frac{d}{\sigma}+\frac{1}{2}\right)^{-1} = \frac{k_d}{\sigma\sqrt{\pi}}\Gamma\left(\frac{d}{\sigma}\right)\Gamma\left(\frac{1}{2}-\frac{d}{\sigma}\right). \tag{5.5c}$$

From equations (5.5) one can see that the singular part of the free energy and the correlation length are universal scaling functions of the variable x_L, i.e.

$$\mathcal{F}_s \equiv \mathcal{F} - \mathcal{F}_0 = L^{-d-z} f_{\mathcal{F}}(x_L) \tag{5.6a}$$

and

$$\xi = L f_\xi(x_L). \tag{5.6b}$$

At the quantum critical point, i.e. $x_L = 0$, the critical amplitudes $f_{\mathcal{F}}(0)$ and $f_\xi(0)$ are dependent upon the geometry of the system and the range of the interaction.

(i) In the region corresponding to $x_L \to \infty$ (in other words, if the quantum parameter r_0 is less than but very close to its critical value r_{0c}), we obtain

$$\xi = \left(2\frac{r_{0c} - r_0}{u_0 D_{d',\sigma}}\right)^{\frac{1}{d_< - d'}} L^{\frac{d-d'}{d_< - d'}}. \tag{5.7}$$

This shows how the correlation length tends to infinity as the size of the system becomes larger. In the bulk system we recover the fact that the correlation length is infinite in the ordered phase in the large-n limit.

(ii) In the case where the quantum parameter r_0 coincides with its critical value r_{0c}, the value of the scaling function $f_\xi(0)$ determines the critical amplitudes at the critical point. The value of these critical amplitudes depends upon the dimension d of the system, the number of infinite sizes d' and the range of the interaction σ. It can be evaluated analytically in the vicinity of the borders (determined by the critical dimensions) where the scaling is valid. The scaling variable vanishes and the function $\Delta_\xi(y_L)$, in equation (5.5b), can be replaced by its asymptotic form for small argument (equation (B.4a)). As a solution for the equation obtained we find

$$\frac{\xi}{L} = \begin{cases} \left(\dfrac{2}{(4\pi)^{\frac{\sigma}{4}}(d - \frac{\sigma}{2})\Gamma(\frac{\sigma}{4}) D_{d',\sigma}}\right)^{\frac{1}{\sigma/2 - d'}} & d - \frac{\sigma}{2} \ll 1 \\ \left(\dfrac{1}{(4\pi)^{\frac{3\sigma}{4}}(\frac{3\sigma}{4} - d)\Gamma(\frac{3\sigma}{4}) D_{d',\sigma}}\right)^{\frac{1}{3\sigma/2 - d'}} & \frac{3\sigma}{2} - d \ll 1. \end{cases} \tag{5.8}$$

In some particular cases where the function $\mathcal{G}_{\alpha,\beta}(x)$ becomes simple, namely in the cases $\sigma = 1$ and $\sigma = 2$ (see appendix A), and for some special cases of the dimensions d and d', it is possible to solve equation (5.5b) numerically. Then one gets

$$\frac{\xi}{L} = \begin{cases} 0.624\,798 & \text{for} \quad d = 1 \quad d' = 0 \quad \sigma = 1 \\ 1.511\,955 & \text{for} \quad d = 2 \quad d' = 0 \quad \sigma = 2 \\ 2\ln\left(\dfrac{1+\sqrt{5}}{2}\right) & \text{for} \quad d = 2 \quad d' = 1 \quad \sigma = 2. \end{cases} \tag{5.9}$$

(iii) The last case we consider here is the one corresponding to the values of the critical quantum parameter smaller that the critical value, i.e. $x_L \to -\infty$. The correlation length is L-independent and is given by

$$\xi = \left(\frac{u_0 D_{d,\sigma}}{2\,(r_{0c} - r_0)}\right)^{\frac{1}{d - d_<}}. \tag{5.10}$$

6. Interplay between quantum and finite-size effects

In this section we consider the case of a finite system at low temperatures. In other words, we will investigate a system confined in the finite geometry of the general form $L^{d-d'} \times \infty^{d'} \times L_\tau^z$ (here we will limit our discussion to the case $d' < d_<$). In this geometry, in addition to the correction terms to equation (3.10b) given in (4.1b) and (5.2d), we also have to add a correction term which accounts for the combined effects of finite sizes and finite temperature. This is given by

$$\Upsilon(\phi, L, T) = \frac{\sqrt{2}L^{-d}}{(2\pi)^{\frac{d+1}{2}}} \sum_m \sum_l{}' \int_0^\infty \frac{\mathrm{d}z}{\sqrt{z}} \exp\left(-z\phi - \frac{T^2 m^2}{4z}\right) |\boldsymbol{l}|^{-d} \Phi_{d/2-1,\sigma}(zL^{-\sigma}|\boldsymbol{l}|^{-\sigma}) \tag{6.1a}$$

where

$$\Phi_{\nu,\sigma}(y) = \int_0^\infty \mathrm{d}x\, x^{\nu+1} J_\nu(x) \mathrm{e}^{-yx^\sigma} \tag{6.1b}$$

was introduced in [22]. In equation (6.1*b*), $J_\nu(x)$ stands for the Bessel function.

The general equation obtained by combining equations (3.10*b*), (4.1*b*), (5.2*d*) and (6.1) can be written in a scaling form whose solution gives the correlation length as a function depending upon two scaling variables. This has the general form

$$\xi = L f_\xi(x_L, LT^{1/z}) = T^{-1/z} f_\tau(x_\tau, LT^{1/z}). \tag{6.2}$$

Actually, we see that there will be competition between the finite sizes and quantum effects depending upon the quantity $LT^{1/z}$, as we will see later. The scaling form (6.2) is in agreement with the predictions of section 2.

The solution of the saddle point equation for a system confined to a general geometry is very difficult to obtain in an explicit form. Even in the two limiting cases of 'low temperature' ($LT^{1/z} \gg 1$) and 'very low temperature' ($LT^{1/z} \ll 1$) the asymptotics of the general equation are very complicated. Nevertheless, these limits provide some useful information about the behaviour of the system in the first or the latter case. Let us mention that the ensuing mathematical equations simplify drastically in the case of short-range forces, i.e. $\sigma = 2$. In this particular case, the analysis of the saddle point equation is identical to the one presented in the framework of the quantum rotor spherical model (for details, see [17]). There is a case when the saddle point equation takes a more simple form, namely the particular case of long-range interaction corresponding to $\sigma = 1$. Even in this case the analysis of the critical behaviour is very complicated.

Before doing this let us present the expressions for the asymptotic forms corresponding to the limiting cases where the finite-temperature effects dominate the finite-size scaling and vice versa.

6.1. Low-temperature regime $LT^{1/z} \gg 1$

In this regime the finite-temperature corrections lead those coming from the finite size of the system. The scaling form of the saddle point equation is given by

$$\begin{aligned} x_\tau = {} & \frac{u_0}{2} D_{d,\sigma} y_\tau^{1/z\nu} + u_0 \frac{k_d}{\sigma\sqrt{\pi}} \Gamma\left(\frac{d}{\sigma}\right) y_\tau^{1/z\nu} \mathcal{K}_2\left(\frac{d}{\sigma} - \frac{1}{2}; \frac{y_\tau}{2}\right) \\ & + \frac{2u_0}{(4\pi)^{d/2}} y_\tau^{2(d/\sigma-1)} \mathcal{K}_\sigma\left(\frac{d}{2} - 1; \frac{y_L}{2}\right) \end{aligned} \tag{6.3a}$$

where we used the functions

$$\mathcal{K}_2(\nu; y) = 2\sum_{m=1}^\infty (my)^{-\nu} K_\nu(2my) \tag{6.3b}$$

and

$$\mathcal{K}_\sigma(\nu; y) = \sum_l{}' \int_0^\infty \mathrm{d}x \frac{x^{\nu+1}}{1 + x^\sigma} \frac{J_\nu(2y\boldsymbol{l}x)}{(y\boldsymbol{l})^\nu}. \tag{6.3c}$$

The latter function has been proposed in [23], where the finite-size scaling properties of the spherical model were discussed. The analytical properties of this function are considered in the same reference.

The function $\mathcal{K}_2(\nu;\, y)$ introduced by equations (6.3) is exponentially decreasing for large values of its argument y (for fixed finite ν). The second function, i.e $\mathcal{K}_\sigma(\nu,\, y)$, is decaying exponentially only in the case of short-range interaction $\sigma = 2$. For the case of long-range interaction corresponding to $\sigma < 2$ its asymptotic form decreases by a power law. The two functions are identical in the case $\sigma = 2$.

Let us recall that the temperature enters in the right-hand side of equation (6.3*a*) through the relation $y_\tau = \sqrt{\phi}/T$. The last term appearing in the scaling form (6.3*a*) is a correction to the bulk system at low temperature resulting from finite-size effects. The temperature-independent part of this term is nothing but the finite-size corrections to the bulk system in its classical limit, i.e. the case when the quantum effects are irrelevant. Now, in the case of short-range interaction ($\sigma = 2$), this equation has been analysed previously [17]. In this case we get *exponential corrections*. In the case of long-range interactions we have *power-law corrections*. This seems to be a general characteristic for systems with long-range interaction (for the case of the spherical model, see [23, 24]).

In the particular case of the two-dimensional system with short-range interaction $d = \sigma = 2$, we get the solution

$$\xi^{-1} \approx \theta T + (2 - d')\sqrt{\frac{2\pi}{5\theta}}\sqrt{\frac{T}{L}}\exp(-TL\theta) \tag{6.4}$$

where $\theta = 0.962\,424$ is a universal constant. This universal form agrees with the predictions made in section 2.

Let us quote here the result for the shift of critical quantum parameter from its bulk value. It is

$$\begin{aligned} r_{0c} - r_{0c}(L, T) &= \frac{2}{\sigma} u_0 k_d T^{(2d/\sigma - 1)} \Gamma\left(\frac{2d}{\sigma} - 1\right) \zeta\left(\frac{2d}{\sigma} - 1\right) \\ &\quad + \frac{u_0}{2T} \frac{\Gamma(d/2 - \sigma/2)}{(4\pi)^{d/2}\Gamma(\sigma/2)} {\sum_{\boldsymbol{l}}}' \left(\frac{|\boldsymbol{l}|L}{2}\right)^{-d+\sigma}. \end{aligned} \tag{6.5}$$

The first term in this equation is the finite-temperature shift of the critical quantum parameter and the second term is nothing but the finite-size shift in the classical limit (i.e. when the quantum fluctuations are negligible in the system) divided by the temperature.

6.2. Very low-temperature regime $LT^{1/z} \ll 1$

Here the finite temperature corrections are negligible in comparison with those coming from the finite-size effects. In this case the scaling form of the saddle point equation turns into

$$x_L = \frac{u_0}{2} D_{d,\sigma} y_L^{d/\sigma - 1/2} + u_0 \Delta_\xi^L(y_L) + u_0 y_L^{d'/\sigma - 1/2} \frac{k_{d'}}{\sigma\sqrt{\pi}} \Gamma\left(\frac{d'}{\sigma}\right) \mathcal{K}_2\left(\frac{d'}{\sigma} - \frac{1}{2};\, \frac{y_L}{2}\right). \tag{6.6}$$

From this equation we can see that the finite-temperature corrections to the finite system at zero temperature are coming mainly from the d' infinite dimensions. In this case the temperature corrections are *exponential*. In the particular case of a two-dimensional system with short-range interaction $d = \sigma = 2$ confined to a strip geometry ($d' = 1$) at the quantum critical point, we get

$$\xi^{-1} \approx \frac{\theta}{L} + \sqrt{\frac{2\pi}{5\theta}}\sqrt{\frac{L}{T}} \exp\left(-\frac{\theta}{TL}\right). \tag{6.7}$$

In the case when the system is confined to a block geometry ($d' = 0$), we have

$$\xi^{-1} \approx \frac{\Omega}{L} + \frac{1}{L}\left[\frac{1}{2\Omega} + \frac{\Omega}{2}\sum_{l}{}^{'}(\Omega^2 + 4\pi^2 \boldsymbol{l}^2)^{-3/2}\right]^{-1} \exp\left(-\frac{\Omega}{LT}\right) \tag{6.8}$$

where $\Omega = 1.511\,955$ is a universal constant. Both results (6.7) and (6.8) are in agreement with the scaling predictions of section 2.

The shift of the critical quantum parameter in this case is given by

$$r_{0c} - r_{0c}(L,T) = \frac{u_0}{2}\frac{\Gamma(d/2-\sigma/4)}{(4\pi)^{d/2}\Gamma(\sigma/4)}\sum_{l}{}^{'}\left(\frac{|\boldsymbol{l}|L}{2}\right)^{-d+\sigma/2}$$
$$+\frac{2}{\sigma}k_{d'}L^{d-d'}T^{(2d'/\sigma-1)}\Gamma\left(\frac{2d'}{\sigma}-1\right)\zeta\left(\frac{2d'}{\sigma}-1\right). \tag{6.9}$$

The first term of the right-hand side of equation (6.9) is discussed in section 5.1. The second term is the correction to the finite-size shift coming from the temperature. It is just the finite temperature shift of a d'-dimensional system multiplied by the volume of a $(d-d')$-finite hypercube with linear size L.

6.3. Short-range case $\sigma = 2$

The equation for the saddle point case reads

$$r_{0c} - r_0 = \frac{u_0}{(4\pi)^{\frac{d+1}{2}}}\Gamma\left(\frac{1-d}{2}\right)\phi^{(d-1)/2} + \frac{\phi^{(d-1)/2}}{(4\pi)^{\frac{d+1}{2}}}\sum_{m,l}{}^{'}\frac{K_{(d-1)/2}[\phi^{1/2}(m^2/T^2 + L^2\boldsymbol{l}^2)]}{[\phi^{1/2}(m^2/T^2 + L^2\boldsymbol{l}^2)]^{(d-1)/2}}. \tag{6.10}$$

This is the simplest form one can get for the saddle point equation. In this case the dynamic critical exponent z, which measures the anisotropy is $z = 1$. The system is symmetric with respect to the change $L \leftrightarrow T^{-1}$.

The general analysis of equation (6.10) follows the one presented in the framework of the quantum rotors model. Here we will not discuss this point since one can obtain the details in [17].

7. Summary

The $\mathcal{O}(n)$ vector φ^4-model is extensively used in the analysis of the critical phenomena because of its direct relevance to the physical reality. In the limit $n \to \infty$ it adds the property of exact solvability at any dimension. This makes it very attractive for the exploration of the the scaling properties of quantum critical phenomena as well as finite-size scaling theory.

In this paper we presented investigations regarding the finite-size scaling of the φ^4-model in the vicinity of its quantum critical point. The striking characteristic of the model is the presence of long-range interaction decaying at large distances r with a power law as $r^{-d-\sigma}$. We considered the model confined to the general geometry of the form $L^{d-d'} \times \infty^{d'} \times L_\tau^z$, where L is the spatial size of the system, $L_\tau \sim T^{-1/z}$ and z is the dynamic critical exponent.

The results were obtained by considering the temperature which governs the crossover between the classical and the quantum critical behaviour as an additional temporal dimension.

A detailed investigation of the alteration of the zero-temperature critical behaviour of the model due to the finite temperature was performed. For dimensions $d_< < d < d_>$, we studied the critical behaviour of the bulk model in the three different regions: classical renormalized, quantum critical and quantum disordered with different behaviours of the correlation length as function of the temperature. The behaviour of the correlation length in these different regions as a function of the dimensionality was calculated. Also, some critical amplitudes were evaluated.

The large-L behaviour and the scaling forms accounting for the finite-size effects were derived for the free energy and the saddle point equation. For the finite-size shift of the the critical quantum parameter, we found $r_{0c} - r_0 \sim L^{-1/\nu}$ in accordance with the general postulates of finite-size scaling. For some particular cases the behaviour of the correlation length and some critical amplitudes were evaluated.

The study of the general case when the system is confined to the general geometry $L^{d-d'} \times \infty^{d'} \times L_\tau^z$, i.e. when the temperature as well as the sizes of the systems are finite turns out to be a very difficult task because of the high anisotropy of the system due to the parameter σ. Nevertheless, one can make interesting deductions in some limiting cases: (i) in the low-temperature regime ($LT^{1/z} \gg 1$), the finite-size corrections to the low-temperature behaviour are *exponentially* small in the case of short-range interaction and are decreasing with a *power law* in the case of long-range interaction; (ii) in the very low-temperature regime ($LT^{1/z} \ll 1$), however, the finite temperature corrections to the finite-size behaviour are always *exponentially* small.

Here, we confined our investigations to the static critical properties of the model, believing that the spherical limit ($n = \infty$) provides a useful tool for studying quantum critical phenomena in dimensions $d > \sigma/2$. The dynamic properties are not well described in this limit and require loop corrections (see, e.g., [25] in the case of the nonlinear quantum sigma model).

Though derived for the special case of the φ^4 model with long-range interaction in the large-n limit, the results obtained here are also expected to hold for many other cases. For example, recently, a model suitable to handle the joint description of classical and quantum fluctuations in an exact manner was considered in a number of publications [20, 26–30]. This model is a modification of the φ^4-lattice model used extensively in the investigation of the critical behaviour of the structural phase transitions [31], in the spirit of the self-consistent phonon approximation method [26].

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Appendix A. Some properties of the function $\mathcal{G}_{\alpha,\beta}(z)$

The functions $\mathcal{G}_{\alpha,\beta}(z)$ were introduced in [20] in order to investigate the zero-temperature finite-size scaling of an anharmonic crystal with long-range interaction at zero temperature. They are entire function series of finite order of growth defined by

$$\mathcal{G}_{\alpha,\beta}(t) = \frac{1}{\sqrt{\pi}} \sum_{k=0}^{\infty} \frac{\Gamma(k+1/2)}{\Gamma(\alpha k + \beta)} \frac{t^k}{k!} \qquad \alpha > 0 \qquad \beta > 0. \tag{A.1}$$

One of the most striking properties of this function is that it obeys the following identity:

$$\frac{1}{\sqrt{1+z}} = \int_0^\infty \mathrm{d}x \mathrm{e}^{-x} x^{\beta-1} \mathcal{G}_{\alpha,\beta}(-x^\alpha z) \tag{A.2}$$

which is obtained by means of term-by-term integration of the series (A.1). The identity (A.2) lies in the basis of the mathematical investigation of finite-size scaling in quantum systems with long-range interaction.

In some particular cases the functions $\mathcal{G}_{\alpha,\beta}(z)$ reduce to known functions. Here we will give some examples which are of interest for us in this paper:

$$\mathcal{G}_{1,1/2}(z) = \frac{\mathrm{e}^z}{\sqrt{\pi}} \tag{A.3a}$$

$$\mathcal{G}_{1/2,1/4}(-z) = \frac{1}{4\sqrt{\pi}} U\left(\frac{3}{4}, \frac{1}{2}, z^2\right) \qquad \text{for} \quad z \geqslant 0 \tag{A.3b}$$

and

$$\mathcal{G}_{1/2,3/4}(-z) = \frac{1}{\sqrt{\pi}} U\left(\frac{1}{4}, \frac{1}{2}, z^2\right) \qquad \text{for} \quad z \geqslant 0 \tag{A.3c}$$

where $U(a, b, z)$ is the confluent hypergeometric function [32].

If we set in the identity (A.2) $z = y^{-\alpha}$, $y > 0$, and $x = ty$, we will obtain the Laplace transform

$$\frac{y^{\alpha/2-\beta}}{(1+y^\alpha)^{1/2}} = \int_0^\infty \mathrm{d}x \mathrm{e}^{-yt} t^{\beta-1} \mathcal{G}_{\alpha,\beta}(-t^\alpha) \tag{A.4}$$

from which we derive a new identity by setting $\beta = \alpha/2$:

$$(1+z^\sigma)^{-1/2} = \int_0^\infty \mathrm{d}x \mathrm{e}^{-xz} x^{\sigma/2-1} \mathcal{G}_{\sigma,\sigma/2}(-x^\sigma). \tag{A.5}$$

With the aid of the last equation we obtained the large-L asymptotic behaviour (5.2*b*) from equation (5.1*b*).

The integral representation of the functions $\mathcal{G}_{\alpha,\beta}(z)$ is obtained with the aid of the Hankel integral for the inverse of the gamma function

$$\frac{1}{\Gamma(z)} = \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{e}^u u^{-z}\, \mathrm{d}z \tag{A.6}$$

where the integration contour C is a loop which starts and ends at $x = -\infty$ and encircles the origin in the positive sense: $-\pi \leqslant \arg z \leqslant \pi$ on C [32]. This enables us to get the result

$$\mathcal{G}_{\alpha,\beta}(z) = \frac{1}{2\pi\mathrm{i}} \int_C \mathrm{d}v \frac{\mathrm{e}^v v^{-\alpha/2+(1-\beta)}}{(v^\alpha - z)^{1/2}}. \tag{A.7}$$

The last result is valid only for $\alpha < 1$. The integrand in (A.7) has a branch point at $v = 0$. A more complete and detailed analysis of the function $\mathcal{G}_{\alpha,\beta}(z)$ will be presented in a subsequent paper. This integral representation is helpful for obtaining the asymptotic behaviour for $z \to \infty$.

In the following we will investigate the asymptotic behaviour of the functions $\mathcal{G}_{\alpha,\beta}(-z)$ for a real argument z. This may be performed by the use of the series

$$\frac{1}{\sqrt{x+t}} = \frac{1}{\sqrt{2x}} \sum_{k=0}^{p} \frac{\Gamma(k+1/2)}{k!} \left(-\frac{t}{x}\right)^k + \frac{1}{\sqrt{2x}} \sum_{k=p}^{\infty} \frac{\Gamma(k+1/2)}{k!} \left(-\frac{t}{x}\right)^k \tag{A.8}$$

for $x \gg t$. The second part of the right-hand side of the last equation is nothing but the expansion in series of the hypergeometric function ${}_2F_1(a, b; c, x)$ multiplied by some coefficients. So, by simple rearrangement, equation (A.8) takes the form

$$\frac{1}{\sqrt{x+t}} = \frac{1}{\sqrt{2x}} \sum_{k=0}^{p} \frac{\Gamma(k+1/2)}{k!} \left(-\frac{t}{x}\right)^k + \frac{1}{\sqrt{\pi x}} \left(-\frac{z}{x}\right)^{p+1} \frac{\Gamma(p+3/2)}{\Gamma(p+2)}\, {}_2F_1\left(1, p+\frac{3}{2}; p+2, -\frac{z}{x}\right). \tag{A.9}$$

Using the integral representation (A.7) and the identity (A.9) we get the asymptotics (for $p \geqslant 1$)

$$\mathcal{G}_{\alpha,\beta}(-x) = \sum_{k=0}^{p}(-1)^k \frac{\Gamma(k+1/2)}{k!\sqrt{\pi}} \frac{x^{-k-1/2}}{\Gamma(\beta-\alpha(k+1/2))} + \mathcal{O}(x^{-p-3/2}) \qquad x \to +\infty. \tag{A.10}$$

In the particular case $\beta = \alpha/2$, the last equation reduces to

$$\mathcal{G}_{\alpha,\alpha/2}(-x) \simeq -\frac{x^{-3/2}}{2\Gamma(-\alpha)}. \tag{A.11}$$

Appendix B. Asymptotics of the function $\Delta_\xi(y)$

Here we present the asymptotic behaviours of the functions $\Delta_\xi(y)$, given in (5.2*d*) for small and large y. These functions are defined by

$$\Delta_\xi(y) = \frac{1}{2}(4\pi)^{-d/2}\sum_{l}{}' \int_0^\infty \mathrm{d}x \exp\left(-\frac{l^2}{4x}\right) x^{\sigma/4-d/2-1}\mathcal{G}_{\sigma/2,\sigma/4}(-x^{\sigma/2}y). \tag{B.1}$$

With the aid of the Jacobi identity for a d-dimensional lattice sum

$$\sum_{l} \mathrm{e}^{-xl^2} = \left(\frac{\pi}{x}\right)^{d/2} \sum_{l} \mathrm{e}^{-\pi^2 l^2/x} \tag{B.2}$$

we transform expression (B.1) into

$$\Delta_\xi(y) = \frac{1}{2}D_{d',\sigma}y^{d'/\sigma-1/2} + \frac{1}{2}\frac{\pi^{\frac{d'}{2}}}{(2\pi)^{\frac{\sigma}{2}}} \int_0^\infty \mathrm{d}x x^{\sigma/4-d'/2-1}\mathcal{G}_{\frac{\sigma}{2},\frac{\sigma}{4}}\left(-y\frac{x^{\frac{\sigma}{2}}}{(2\pi)^\sigma}\right)$$
$$\times\left[\sum_{l}{}'\mathrm{e}^{-xl^2} - \left(\frac{\pi}{x}\right)^{d/2}\right]. \tag{B.3}$$

In order to be able to get a reasonable expression for the integral in the last equation we have to avoid the divergence in the square brackets. To this end we add and subtract from the function $\mathcal{G}_{\alpha,\beta}(x)$ its small asymptotic behaviour to the first order (see appendix A), which enables us to write (after some algebra)

$$\Delta_\xi(y) = \frac{1}{2}D_{d',\sigma}y^{d'/\sigma-1/2} - \frac{1}{2}D_{d,\sigma}y^{d/\sigma-1/2} + \mathcal{W}_{d,d',\sigma}(y) + \frac{1}{(2\pi)^\sigma}\frac{\pi^{d'/2}}{\Gamma(\sigma/4)}\mathcal{C}_{d,d',\sigma} \tag{B.4a}$$

where we used the notations

$$\mathcal{W}_{d,d',\sigma} = \frac{\pi^{d'/2}}{(2\pi)^\sigma}\sum_{l}{}' \int_0^\infty \mathrm{d}x x^{\sigma/4-d'/2-1}\mathrm{e}^{-xl^2}\left[\mathcal{G}_{\sigma/2,\sigma/4}\left(-\frac{x^{\sigma/2}y}{(2\pi)^\sigma}\right) - \frac{1}{\Gamma(\sigma/4)}\right] \tag{B.4b}$$

$$\begin{aligned}\mathcal{C}_{d,d',\sigma} &= \sum_{l}{}' \int_0^\infty \mathrm{d}x x^{\sigma/4-d'/2-1}\mathrm{e}^{-xl^2} - \pi^{(d-d')/2}\int_0^\infty \mathrm{d}x x^{\sigma/4-d'/2-1} \\ &= \lim_{\lambda\to 0}\left\{\sum_{l}{}'\Gamma\left(\frac{\sigma}{4}-\frac{d'}{2},\lambda l^2\right)|l|^{d'/2-\sigma/4}\right. \\ &\qquad \left. - \int_{-\infty}^\infty\cdots\int_{-\infty}^\infty \mathrm{d}x\Gamma\left(\frac{\sigma}{4}-\frac{d'}{2},\lambda l^2\right)|l|^{\frac{d'}{2}-\frac{\sigma}{4}}\right\}.\end{aligned} \tag{B.4c}$$

Here $\Gamma(\alpha, x)$ is the incomplete gamma function. The expression in equation (B.4*c*) is a generalization of the Madelung-type constant, and is y-independent. Indeed, this equation defines the finite-size shift of the critical quantum parameter r_{0c} at zero temperature. It is easy (following [17]) to show that equations (B.4*c*) and (5.3) are equivalent.

For small y the asymptotic behaviour of $\Delta_\xi(y)$ is given by the first term in the right-hand side of equation (B.4*a*).

For large y the asymptotics of the function $\Delta_\xi(y)$ are obtained by substituting the large-x behaviour of the functions $\mathcal{G}_{\alpha,\beta}(x)$ from equation (A.10) in definition (B.1). After some calculations one ends up with

$$\Delta_\xi(y) \simeq -\frac{1}{4}y^{-3/2}\frac{(4\pi)^{\sigma/2}}{\Gamma(-\sigma/2)}\Gamma\left(\frac{d+\sigma}{2}\right)\sum_{l}\left(\frac{1}{\pi|l|}\right)^{\frac{d+\sigma}{2}}. \tag{B.5}$$

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